Fourier Volatility Estimation Method: Theory and Applications with High Frequency

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Motivation

- Computation of **volatility/covariance** of financial asset returns plays a central role for many issues in finance: **risk management**, hedging strategies, forecasting...
- Black&Scholes model - **constant volatility** - does not account for: heteroschedasticity, predictability, volatility smile, covariance between asset returns and volatility (leverage effect) $\Rrightarrow$
  stochastic volatility models proposed to model asset price evolution and to price options (adding **risk factors** represented by Brownian motions [Heston, 1993, Hull and White, 1987, Stein and Stein, 1991], **jumps** [Bates, 1996], or introducing **memory** [Hobson and Rogers, 1998])
- Availability of **high frequency data** have the potential to improve the capability of computing volatility/covariances in an efficient way to many extend [Andersen et al., 2006] (forecasting), [Bollerslev and Zhang, 2003] (risk factor models), [Fleming et al., 2003] (asset allocation)....

Outline

- Definition of Fourier estimator of spot and integrated volatility/covariance
- Properties of Fourier estimator with high frequency data
- Potentiality of Fourier estimator for some applications:
    - Volatility of Volatility and Leverage estimation
    - Forecasting Volatility

Non-parametric and model free context

Model: continuous Brownian semimartingale

$$(\mathbf{B}) \qquad dp^j(t) = \sum_{i=1}^{d} \sigma_i^j(t)\, dW^i + b^j(t)\, dt, \qquad j = 1, \dots, n,$$

$W = (W^1, \dots, W^d)$ are independent Brownian motions and σ_*^* and b^* are adapted random processes satisfying

$$E[\int_0^{2\pi} (b^j(t))^2 dt] < \infty, \quad E[\int_0^{2\pi} (\sigma_i^j(t))^4 dt] < \infty \qquad i = 1, \dots, d,\ j = 1, \dots, m$$

Objective: estimation of the time dependent *volatility matrix*:

$$\Sigma^{jk}(t) = \sum_{i=1}^{d} \sigma_i^j(t) \sigma_i^k(t) \qquad j, k = 1, \dots, n$$

Main Issues

$p^*(t)$ asset log-price Brownian semimartingale $\Rightarrow$ integrated volatility/covariance

$$\int_0^t \Sigma^{ik}(s)ds = P-\lim_{n\to\infty} \sum_{0\le j<t2^n} \left(p^i((j+1)2^{-n}) - p^i(j2^{-n})\right)\left(p^k((j+1)2^{-n}) - p^k(j2^{-n})\right).$$

Nevertheless, when sampling high frequency returns, three **difficulties** arise:

1) the distortion from efficient prices due to the **market microstructure** noise such as price discreteness, infrequent trading,...[Roll, 1984].
2) instantaneous volatility computation involves a sort of **numerical derivative**, which gives rise to numerical instabilities [Foster and Nelson, 1996, Comte and Renault, 1998]

In the multivariate case also:

3) the **non-synchronicity** of the arrival times of trades across markets leads to a bias towards zero in correlations among stocks as the sampling frequency increases [Epps, 1979]

Mean covariance [Malliavin and M. 2002, 2009]

Theorem

Consider a process p satisfying the assumption **(B)**. Then we have:

$$\frac{1}{2\pi}\mathcal{F}(\Sigma^{ij}) = \mathcal{F}(dp^i) *_B \mathcal{F}(dp^j). \tag{1}$$

The convergence of the convolution product (1) is attained in probability

where, for $k \in \mathbf{Z}$

$$\mathcal{F}(dp^i)(k) := \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}kt}\, dp^i(t)$$

$$(\Phi *_B \Psi)(k) := \lim_{N\to\infty} \frac{1}{2N+1}\sum_{s=-N}^{N} \Phi(s)\Psi(k-s)$$

$$\mathcal{F}(\Sigma^{ij})(k) := \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}kt}\, \Sigma^{ij}(t)\, dt$$

Fourier instantaneous covariance computation

By the theorem we gather all the Fourier coefficients of the volatility matrix by means of the Fourier transform of the log-returns. Then reconstruct the **co-volatility functions** $\Sigma^{ij}(t)$ from its Fourier coefficients by the Fourier-Fejer summation:
let for $i, j = 1, 2$ and for any $|k| \leq N$,

$$c_N^{ij}(k) := \frac{1}{2N+1} \sum_{|s| \leq N} \mathcal{F}(dp^i)(s)\mathcal{F}(dp^j)(k-s),$$

then

$$\Sigma^{ij}(t) = \lim_{N \to \infty} \sum_{|k|<N} (1 - \frac{|k|}{N}) c_N^{ij}(k) e^{ikt}$$

Consistency

Given observation times $(t_i^1)_{0\leq i\leq n_1}$ and $(t_j^2)_{0\leq j\leq n_2}$, $\rho(n) := \rho^1(n_1) \vee \rho^2(n_2)$ and $\rho^*(n_*) = \max_{t_l^*} |t_{l+1}^* - t_l^*|$, define:

$$c_k(dp^1_{n_1}) := \frac{1}{2\pi} \sum_{i=0}^{n_1-1} e^{-\mathrm{i}kt_i^1}(p^1(t^1_{i+1}) - p^1(t^1_i))$$

$$c_k(dp^2_{n_2}) := \frac{1}{2\pi} \sum_{j=0}^{n_2-1} e^{-\mathrm{i}kt_j^2}(p^2(t^2_{j+1}) - p^2(t^2_j))$$

$$c_k(\Sigma^{12}) := \frac{1}{2\pi} \int_0^{2\pi} e^{-\mathrm{i}kt}\Sigma^{12}(t)dt$$

Consistency

Define for any $|k| \leq N$

$$\alpha_k(N, p^1_{n_1}, p^2_{n_2}) = \frac{2\pi}{2N+1} \sum_{|s| \leq N} c_s(dp^1_{n_1}) c_{k-s}(dp^2_{n_2}). \tag{2}$$

Suppose that $N\rho(n) \to 0$ as $N, n \to \infty$. Then, for any k, in probability

$$\alpha_k(N, p^1_{n_1}, p^2_{n_2}) \to c_k(\Sigma^{12})$$

In probability, uniformly in t,

$$\widehat{\Sigma}^{12}_{n_1,n_2,N}(t) := \sum_{|k| \leq N} (1 - \frac{|k|}{N}) \alpha_k(N, p^1_{n_1}, p^2_{n_2}) e^{ikt} \to \Sigma^{12}(t) \tag{3}$$

Asymptotic Normality

Suppose and $\rho(n)N^{4/3} \rightarrow 0$, $\rho(n)N^{2\alpha} \rightarrow \infty$, if $\alpha > \frac{2}{3}$ and assumption **(A)** holds. Then for any function $g \in Lip(\alpha)$, with compact support in $(0, 2\pi)$,

$$(\rho(n))^{-\frac{1}{2}} \int_0^{2\pi} g(t)(\widehat{\Sigma}^{12}_{n,N}(t) - \Sigma^{12}(t))dt$$

converges in law to a mixture of Gaussian distribution with variance

$$\int_0^{2\pi} H'(t)g^2(t)(\Sigma^{11}(t)\Sigma^{22}(t) + (\Sigma^{12}(t))^2)dt.$$

(A) $H(t)$ quadratic variation of time
(i) $\rho(n) \rightarrow 0$ and $n_i\rho(n) = 0(1)$ for $i = 1, 2$
(ii) $H_n(t) := \frac{n}{2\pi} \sum_{t^1_{i+1} \wedge t^2_{j+1} \leq t} (t^1_{i+1} \wedge t^2_{j+1} - t^1_i \vee t^2_j)^2 I_{\{t^1_i \vee t^2_j < t^1_{i+1} \wedge t^2_{j+1}\}} \rightarrow H(t)$ as $n \rightarrow \infty$
(iii) $H(t)$ is continuously differentiable
If data are synchronous and equally spaced then $H'(t) = 1$, [Mykland and Zhang, 2006]

Spot volatility estimators

Alternative estimators of **spot volatility**, NOT involving numerical derivative:

[Genon-Catalot, Laredo and Picard, 1992],
[Hoffman, Munk and Schmidt-Hieber, 2010]
[Ogawa and Sanfelici, 2011], [Ogawa and Ngo, 2010]
[Mancini, Mattiussi and Reno, 2012]

Model with microstructure

Consider the following model for the observed log-returns

$$\widetilde{p}^i(t) := p^i(t) + \eta^i(t) \quad \text{for } i = 1, 2,$$

Moreover the following assumptions hold:

(M)

M1. $p := (p^1, p^2)$ and $\eta := (\eta^1, \eta^2)$ are independent processes, moreover $\eta(t)$ and $\eta(s)$ are independent for $s \neq t$ and $E[\eta(t)] = 0$ for any t.
M2. $E[\eta^i(t)\eta^j(t)] = \omega_{ij} < \infty$ for any t, $i, j = 1, 2$.

or **(MD)**

the microstructure noise is correlated with the price process and there is also a temporal dependence in the noise components

Fourier estimator of integrated covariance

$$\widehat{\Sigma}^{12}_{N,n_1,n_2} := \frac{(2\pi)^2}{2N+1} \sum_{|s|\leq N} c_s(dp^1_{n_1}) c_{-s}(dp^2_{n_2})$$

If $\rho(n)N \to 0$, the following convergence in probability holds:

$$\lim_{n_1,n_2,N\to\infty} \widehat{\Sigma}^{12}_{N,n_1,n_2} = \int_0^{2\pi} \Sigma^{12}(t)dt.$$

In the application we consider also the following version which preserves **definite positiveness of the covariance matrix**

$$\widehat{\Sigma}^{12}_{N,n_1,n_2} := \frac{(2\pi)^2}{N+1} \sum_{|s|\leq N} (1 - \frac{|s|}{N}) c_s(dp^1_{n_1}) c_{-s}(dp^2_{n_2}).$$

Quadratic covariation type estimators

Estimators based on the choice of a synchronization procedure, which gives the observations times $\{0 = \tau_1 \leq \tau_2 \leq \cdots \leq \tau_n \leq 2\pi\}$ for both assets

Realized covariation $$RC^{12} := \sum_{i=1}^{n-1} \delta_i(p^1)\delta_i(p^2),$$

Realized covariation with leads and lags $$RCLL^{12} := \sum_i \sum_{h=-l}^{L} \delta_{i+h}(p^1)\delta_i(p^2),$$

Realized covariance kernels estimator $$RCLLW^{12} := \sum_i \sum_{h=-l}^{L} w(h)\delta_{i+h}(p^1)\delta_i(p^2),$$

where $\delta_i(p^*) = p^*(\tau_{i+1}) - p^*(\tau_i)$, and $w(h)$ is a kernel.

inconsistent for asynchronous observations and **inconsistent under (i.i.d) noise**, the MSE diverges as the number of observations increases; $RCLL^{1,2}$, $RCLLW^{1,2}$ more robust to microstructure noise, but they are much biased by dependent noise contaminations [Griffin and Oomen, 2010]

Refresh times consistent estimators

• [Barndorff-Nielsen and al., 2008a] Realized covariance kernels with refresh times consistent for asynchronous observations/robust to some kind of noise

$$K^{12} := \sum_{h=-n}^{n} k\left(\frac{h}{H+1}\right) \Gamma_h^{12},$$

Γ_h^{12} is h-th realised autocovariance of the two assets, $k(\cdot)$ belongs to a suitable class of kernel functions (Parzen).

• [Kinnebrock and Podolskij, 2008] Modulated Realised Covariation
pre-averaging technique to reduce the microstructure effects (if one averages a number of observed log-prices, one is closer to the latent process $p(t)$)

Consistent estimators

- [Hayashi and Yoshida, 2005] All-overlapping estimator

$$AO^{12} := \sum_{i,j} \delta_{I_i^1}(p^1)\delta_{I_j^2}(p^2) I_{(I_i^1 \cap I_j^2 \neq \emptyset)},$$

where $\delta_{I_i^*}(p^*) := p^*(t_{i+1}^*) - p^*(t_i^*)$. Consistent for asynchronous observations, but NOT robust to noise: $\Rrightarrow$

- [Voev et Lunde, 2007] Sub-sampled All-overlapping estimator
- [Christensen, Podolskij and Vetter, 2012] Pre-averaged All-overlapping estimator

MSE

regular asynchronous trading: the asset 1 trades at regular points: $\Pi^1 = \{t_i^1 : i = 1, \dots, n_1$ and $t_{i+1}^1 - t_i^1 = \frac{2\pi}{n_1}\}$; also asset 2 trades at regular points: $\Pi^2 = \{t_j^2 : j = 1, \dots, n_2$ and $t_{j+1}^2 - t_j^2 = \frac{4\pi}{n_1}\}$, but no trade of asset 1 occurs at the same time of a trade of asset 2

$$MSE_{AOm} = o(1) + 2\omega_{11} \sum_{j=1}^{\frac{n}{2}-1} E[\int_{t_j^2}^{t_{j+1}^2} \Sigma^{22}(t)dt] + 2\omega_{22} \sum_{i=1}^{n-1} E[\int_{t_i^1}^{t_{i+1}^1} \Sigma^{11}(t)dt] + \\ +2(n-1)\omega_{11}\omega_{22}$$

$$MSE_{Fm} = o(1) + 2\omega_{11} \sum_{j=1}^{\frac{n}{2}-1} D_N^2(t_{n-1}^1 - t_j^2) E[\int_{t_j^2}^{t_{j+1}^2} \Sigma^{22}(t)dt] + \\ +2\omega_{22} \sum_{i=1}^{n-1} D_N^2(t_i^1 - t_{\frac{n}{2}-1}^2) E[\int_{t_i^1}^{t_{i+1}^1} \Sigma^{11}(t)dt] + 4\omega_{11}\omega_{22} D_N^2(t_{n-1}^1 - t_{\frac{n}{2}-1}^2)$$

where $D_N(t) := \frac{1}{2N+1} \frac{\sin[(N+\frac{1}{2})t]}{\sin \frac{t}{2}}$

Montecarlo Analysis

We simulate discrete data from the continuous time bivariate GARCH model

$$\begin{bmatrix} dp^1(t) \\ dp^2(t) \end{bmatrix} = \begin{bmatrix} \beta_1\sigma_1^2(t) \\ \beta_2\sigma_4^2(t) \end{bmatrix} dt + \begin{bmatrix} \sigma_1(t) & \sigma_2(t) \\ \sigma_3(t) & \sigma_4(t) \end{bmatrix} \begin{bmatrix} dW_5(t) \\ dW_6(t) \end{bmatrix}$$

$$d\sigma_i^2(t) = (\omega_i - \theta_i\sigma_i^2(t))dt + \alpha_i\sigma_i^2(t)dW_i(t), \quad i = 1, \ldots, 4,$$

The logarithmic noises $\eta^1(t), \eta^2(t)$ are i.i.d. Gaussian, possibly contemporaneously correlated.

We generate second-by-second return and variance paths over a daily trading period of $h = 6$ hours. Then we sample the observations according to different scenarios: *regular synchronous trading* with durations $\rho_1 = \rho(n_1)$ and $\rho_2 = 2\rho_1$; *regular non-synchronous trading* with durations ρ_1 and $\rho_2 = 2\rho_1$ and displacement $\delta \cdot \rho_1$; *Poisson trading* with durations between trades drawn from an exponential distribution with means λ_1, λ_2.

	Reg-NS		Reg-S + Unc		Reg-NS + Unc		Reg-NS + Cor	
	MSE	bias	MSE	bias	MSE	bias	MSE	bias
$\hat{\Sigma}^{12}_{N,n_1,n_2}$	5.72e-4	-9.88e-3	3.35e-4	-6.09e-3	7.29e-4	-1.12e-2	4.73e-4	-8.82e-3
$RC^{12}_{0,5min}$	2.96e-2	-1.68e-1	1.06e-3	8.80e-4	3.45e-2	-1.80e-1	3.20e-2	-1.74e-1
RC^{12}_{1min}	9.14e-3	-8.44e-2	2.08e-3	2.70e-3	1.12e-2	-9.16e-2	9.74e-3	-8.65e-2
RC^{12}_{5min}	1.16e-2	-1.80e-2	1.14e-2	5.00e-3	1.44e-2	-2.33e-2	1.13e-2	-1.68e-2
$RCLL^{12}_{0,5min}$	2.88e-3	-1.68e-3	3.34e-3	2.94e-3	3.71e-3	-2.43e-3	3.15e-3	-1.55e-3
$RCLL^{12}_{1min}$	6.40e-3	-3.13e-3	6.42e-3	5.04e-3	8.00e-3	-3.37e-4	6.13e-3	3.09e-3
$RCLL^{12}_{5min}$	3.35e-2	1.11e-2	3.12e-2	3.15e-4	4.23e-2	-7.22e-3	3.61e-2	6.79e-3
AO^{12}	4.72e-4	-1.20e-3	4.47e-4	-1.08e-3	6.88e-4	9.45e-4	5.98e-4	-5.91e-4
K^{12}	9.33e-4	-8.13e-3	9.13e-4	-5.22e-4	1.28e-3	-6.32e-3	1.09e-3	-7.18e-3
MRC^{12}	2.80e-3	-3.27e-2	2.57e-3	-2.55e-2	3.38e-3	-3.01e-2	2.91e-3	-2.87e-2

	Reg-NS + Dep		Poisson + Unc		Poisson + Cor		Poisson + Dep	
	MSE	bias	MSE	bias	MSE	bias	MSE	bias
$\hat{\Sigma}^{12}_{N,n_1,n_2}$	3.96e-4	-6.32e-3	1.07e-3	-1.38e-2	1.18e-3	-1.53e-2	1.00e-3	-1.43e-2
$RC^{12}_{0,5min}$	3.02e-2	-1.66e-1	3.33e-2	-1.76e-1	3.11e-2	-1.70e-1	2.91e-2	-1.64e-1
RC^{12}_{1min}	9.97e-3	-8.17e-2	1.08e-2	-8.95e-2	1.05e-2	-8.85e-2	1.03e-2	-8.62e-2
RC^{12}_{5min}	1.47e-2	-1.70e-2	1.28e-2	-2.50e-2	1.36e-2	-2.06e-2	1.23e-2	-2.64e-2
$RCLL^{12}_{0,5min}$	4.42e-3	3.20e-3	3.81e-3	-7.98e-3	3.40e-3	-6.84e-3	3.73e-3	-9.08e-3
$RCLL^{12}_{1min}$	8.06e-3	-9.21e-4	6.81e-3	-3.41e-3	7.23e-3	1.26e-3	7.80e-3	3.78e-3
$RCLL^{12}_{5min}$	3.59e-2	-1.60e-2	3.31e-2	-3.59e-3	3.74e-2	6.35e-3	3.67e-2	-1.47e-2
AO^{12}	7.42e-3	7.46e-2	1.29e-3	-8.75e-4	1.24e-3	9.32e-3	8.10e-3	7.49e-2
K^{12}	5.25e-3	5.43e-2	5.88e-3	-6.35e-2	4.57e-3	-5.46e-2	2.85e-3	-1.95e-2
MRC^{12}	3.93e-3	-1.59e-2	4.19e-3	-3.00e-2	3.71e-3	-2.71e-2	4.72e-3	-2.24e-2

Tabella: Comparison of integrated volatility estimators. The noise variance is 90% of the total variance for 1 second returns. $\rho_1 = 5$ sec, $\rho_2 = 10$ sec with a displacement of 0 seconds for Reg-S and 2 seconds for Reg-NS trading; $\lambda_1 = 5$ sec and $\lambda_2 = 10$ sec for Poisson trading.

	Reg-S + Unc		Reg-NS + Unc		Reg-NS + Cor		Reg-NS + Dep	
	MSE	bias	MSE	bias	MSE	bias	MSE	bias
$\hat{\Sigma}^{12}_{N,n_1,n_2}$	3.42e-4	-3.28e-3	3.93e-4	-4.93e-3	4.37e-4	-3.86e-3	8.67e-4	-4.90e-3
$RC^{12}_{0,5min}$	3.81e-2	4.01e-3	6.92e-2	-1.66e-1	8.73e-2	-1.81e-1	2.00e+0	-1.47e-1
RC^{12}_{1min}	2.26e-2	-4.08e-3	3.35e-2	-8.09e-2	4.31e-2	-8.67e-2	1.14e+0	-1.19e-1
RC^{12}_{5min}	1.93e-2	-4.05e-3	2.21e-2	-1.48e-2	2.67e-2	-8.87e-3	2.84e-1	-5.89e-2
$RCLL^{12}_{0,5min}$	2.77e-2	5.92e-3	3.46e-2	-1.57e-3	4.28e-2	2.48e-3	1.37e+0	-3.36e-2
$RCLL^{12}_{1min}$	2.29e-2	-1.27e-3	2.59e-2	-9.86e-4	3.45e-2	-8.57e-3	6.82e-1	1.37e-2
$RCLL^{12}_{5min}$	4.47e-2	1.02e-3	4.46e-2	1.02e-3	4.91e-2	1.48e-2	2.22e-1	-6.84e-4
AO^{12}	9.76e-2	5.38e-3	7.71e-2	2.49e-2	9.23e-2	-7.94e-3	4.40e+0	-8.95e-3
K^{12}	3.69e-2	-2.57e-3	3.80e-2	1.67e-2	4.94e-2	-7.48e-3	2.14e+0	2.44e-2
MRC^{12}	6.42e-3	-1.66e-2	7.74e-3	-1.40e-2	8.04e-3	-9.84e-3	1.25e-2	-2.21e-2

	Poisson + Unc		Poisson + Cor		Poisson + Dep			
	MSE	bias	MSE	bias	MSE	bias		
$\hat{\Sigma}^{12}_{N,n_1,n_2}$	1.14e-3	-1.26e-2	5.35e-4	-5.62e-3	5.24e-4	-3.54e-3		
$RC^{12}_{0,5min}$	9.50e-2	-2.10e-1	5.11e-2	-4.78e-2	1.82e+0	-1.44e-1		
RC^{12}_{1min}	4.71e-2	-1.04e-1	3.00e-2	-1.54e-2	1.03e+0	-6.62e-2		
RC^{12}_{5min}	2.79e-2	-3.07e-2	2.39e-2	-1.75e-2	3.01e-1	-3.93e-2		
$RCLL^{12}_{0,5min}$	4.13e-2	-1.00e-2	3.70e-2	3.25e-4	1.43e+0	6.61e-2		
$RCLL^{12}_{1min}$	3.18e-2	1.08e-2	2.87e-2	-8.09e-3	6.96e-1	-3.81e-2		
$RCLL^{12}_{5min}$	5.88e-2	1.61e-2	4.39e-2	-2.27.e-3	2.40e-1	-3.03e-2		
AO^{12}	8.83e-2	5.85e-3	1.27e+0	1.07e+0	2.91e+0	1.12e-1		
K^{12}	4.87e-2	-5.59e-2	2.63e-1	4.70e-1	1.61e+0	1.83e-3		
MRC^{12}	1.23e-2	-2.12e-2	9.94e-3	-2.22e-2	1.58e-2	-2.66e-2		

Tabella: Comparison of integrated volatility estimators. Increased Noise (as in [Griffin and Oomen, 2010]). $\rho_1 = 5$ sec, $\rho_2 = 10$ sec with a displacement of 0 seconds for Reg-S and 2 seconds for Reg-NS trading; $\lambda_1 = 5$ sec and $\lambda_2 = 10$ sec for Poisson trading.

Feasible estimators

In order to produce **feasible central limit theorems** for all the estimators, and as a consequence feasible confidence intervals, it is necessary to obtain **efficient estimators of the so called quarticity**, which appears as conditional variance in the central limit theorems.

Nevertheless, the studies about estimation of quarticity are still few:

> estimating integrated quarticity reasonably efficiently is a tougher problem than estimating the integrated volatility, as the effect of noise is magnified up

[Barndorff-Nielsen and al., 2008a]

Fourier Quarticity estimator

$$\sigma^4_{n,N,M} := 2\pi \sum_{|s|<M} c_s(\sigma^2_{n,N}) c_{-s}(\sigma^2_{n,N})$$

[M. and Sanfelici, 2012]: effectiveness of Fourier estimation method when applied to compute the quarticity in the presence of microstructure noise, due to the intrinsic robustness of the Fourier estimator of volatility

Fourier estimator properties

1) uses all the available observations, no synchronization of the original data: it is based on the **integration of the time series of returns** rather than on its differentiation
2) it is designed specifically **for high frequency data**: by cutting the highest frequencies, it uses as much as possible of the sample path without being more sensitive to market frictions

Focus

3) it allows to reconstruct the volatility/covariance as a **stochastic function of time**: we can handle the volatility function as an observable variable

Stochastic Volatility Model

$$\begin{cases} dp(t) = \sigma(t)dW_0(t) + a(t)dt \\ dv(t) = \gamma(t)dZ(t) + b(t)dt \end{cases}$$

$v(t) := \sigma^2(t)$ is the variance process,
W_0 and Z correlated Brownian motions: $\eta(t)dt = dW_0(t) * dZ(t)$

Compute pathwise the diffusion coefficients $\sigma(t)$, $\gamma(t)$ and the covariance between the price and the instantaneous variance, $\varrho(t)$, given the observation of the asset price trajectory $p(t)$, $t \in [0, T]$

Method

Compute pathwise the diffusion coefficients $\sigma(t)$, $\gamma(t)$ and the covariance between the price and the instantaneous variance, $\varrho(t)$, given the observation of the asset price trajectory $p(t),\ t \in [0, T]$

1. compute the Fourier coefficients of the unobservable instantaneous variance process $v(t),\ t \in [0, T]$ in terms of the Fourier coefficients of $p(t) \Rrightarrow v(t)$ is reconstructed from its Fourier coefficients by the Fourier-Fejer summation method

2. the instantaneous variance $v(t)$ is handled as an observable variable $\Rrightarrow$ we iterate the procedure to compute the volatility of the variance process identifying the two components: volatility of variance ($\gamma(t)$) and asset price-variance covariance ($\varrho(t)$)

3. finally compute $\eta(t)$ by to the identity $\varrho(t) = \eta(t)\sigma(t)\gamma(t)$ with $\sigma(t)$ and $\gamma(t)$ a.s. positive

Volatility of Volatility

- Derive an estimator for Fourier coefficients $(c_k(\gamma^2))$ of $\gamma^2(t)$ given the observations of the variance process:
  By parts
  $$c_k(dv_{n,M}) = \mathrm{i}k c_k(v_{n,M}) + \frac{1}{2\pi}(v_{n,M}(2\pi) - v_{n,M}(0)),$$
  where $c_k(v_{n,M})$ were computed from dp
- Let

$$c_k(\gamma^2_{n,N,M}) := \frac{2\pi}{2N+1} \sum_{|j|\leq N} c_j(dv_{n,M}) c_{k-j}(dv_{n,M})$$

- If $\frac{N^4}{M} \to 0$ and $M^{\frac{5}{4}}\rho(n) \to 0$ for $n, N, M \to \infty$

$$P - \lim_{n,N,M\to\infty} c_k(\gamma^2_{n,N,M}) = c_k(\gamma^2)$$

Leverage

To compute the instantaneous covariance $\varrho(t)$ we exploit the **multivariate version of Fourier estimator**

- obtain a consistent estimator of the k-th Fourier coefficient of $\varrho(t)$ starting from the Fourier coefficients of the observed asset returns

$$c_k(\varrho_{n,N,M}) = \frac{2\pi}{2N+1} \sum_{|j| \leq N} c_j(dp_n) c_{k-j}(dv_{n,M})$$

- If $\frac{N^2}{M} \to 0$ and $M\rho(n) \to 0$ for $n, N, M \to \infty$, then

$$P - \lim_{n,N,M\to\infty} c_k(\varrho_{n,N,M}) = c_k(\varrho)$$

(Preliminary) Montecarlo Analysis

Replicate numerical experiment by [Bollerslev and Zhou, 2002] who apply a **generalized moment method (GMM)** exploiting high frequency data, to estimate ξ, $\xi\eta(=\varrho)$ and square root process:

$$dp(t) = \sqrt{v(t)}dW_0(t)$$

$$dv(t) = k(\theta - v(t))dt + \xi\sqrt{v(t)}dZ(t)$$

k=mean reversion
θ=long run
ξ= volatility of variance

W_0, Z are standard Brownian motions $dW_0(t) * dZ(t) = \eta dt$

Montecarlo Analysis

We consider three parameter scenarios suggested in [Bollerslev and Zhou, 2002]:

$$\text{Scenario A}: \quad \text{k} = 0.03, \ \theta = 0.25, \ \xi = 0.1,$$

$$\text{Scenario B}: \quad \text{k} = 0.1, \ \theta = 0.25, \ \xi = 0.1,$$

$$\text{Scenario C}: \quad \text{k} = 0.1, \ \theta = 0.25, \ \xi = 0.2,$$

Two values of η: $\eta = -0.2$ and $\eta = -0.7$

True values	Mean		Median		Standard Deviation	
	T=1000	T=4000	T=1000	T=4000	T=1000	T=4000
Panel A						
$\xi\eta = -0.02$	-0.0220	-0.0221	-0.0125	-0.0262	0.2157	0.1474
$\xi = 0.1$	0.1040	0.1014	0.1040	0.1014	0.0890	0.0768
Panel A						
$\xi\eta = -0.07$	-0.0706	-0.0729	-0.0622	-0.0730	0.2201	0.2106
$\xi = 0.1$	0.1075	0.1048	0.1075	0.1048	0.0856	0.0138
Panel B						
$\xi\eta = -0.02$	-0.0181	-0.0282	-0.0177	-0.0201	0.2865	0.2488
$\xi = 0.1$	0.1012	0.1069	0.1012	0.1069	0.0699	0.0695
Panel B						
$\xi\eta = -0.07$	-0.0717	-0.0737	-0.1314	-0.0711	0.2828	0.2560
$\xi = 0.1$	0.1330	0.1075	0.1331	0.1075	0.1188	0.0753
Panel C						
$\xi\eta = -0.04$	-0.0469	-0.0409	-0.1394	-0.0373	0.2707	0.1987
$\xi = 0.2$	0.2023	0.2066	0.2341	0.2165	0.1474	0.0892
Panel C						
$\xi\eta = -0.14$	-0.1263	-0.1569	-0.1442	-0.1561	0.3380	0.0616
$\xi = 0.2$	0.1994	0.2006	0.1984	0.2130	0.1571	0.0926

Tabella: Average value, median value and standard deviation of ξ and of $\xi\eta$ for three parameter scenarios, two correlation values and two choices of the size of the simulation sample.

Simulation results are satisfactory. The mean and the median of the parameters obtained in Table 3 are similar to those obtained in [Bollerslev and Zhou, 2002], only the standard deviation is slightly higher.

Note: the methodology in [Bollerslev and Zhou, 2002] exploits the knowledge of the square root model that generates the asset price observations, our methodology instead is model free and is able to recover the parameters of the data generating process without making a parametric assumption.

The performance of Fourier method is comparable to the one of the parametric method proposed in [Bollerslev and Zhou, 2002].
This exercise is only an illustrative example to show the efficiency of the method: as a matter of fact, parametric methods exploiting the assumption of a model, are expected to outperform non parametric methods.
Further analysis on going, where microstructure contamination is included.

Forecasting volatility

- Empirical analysis have shown that the forecasting performance of the realized volatility is superior to that of classical ARCH models [Andersen and al., 2003]
- BUT microstructure noise contamination badly affects realized volatility's forecasting performance of future integrated volatility
- [Andersen et al., 2006] extend the analysis to the case where the price is contaminated by microstructure noise, using other realized volatility measures that are corrected to noise

Methodology

The forecasting performance of the Fourier estimator of integrated volatility is analyzed in [Barucci, Magno and M., 2010]:

Given a measure of the integrated volatility in the period $[t-1, t]$ we evaluate its capability of forecasting the integrated volatility on day $[t, t+1]$:
to this end the linear regression of the one day ahead integrated volatility over the today volatility is considered: the forecasting performance can be evaluated through the R^2 (coefficient of determination) of the linear regression:
the choice of the R^2 implicitly corresponds to use the mean squared forecast error which is the most common benchmark to evaluate the forecasting performance.

Model under microstructure

The logarithm of the observed asset price

$$\widetilde{p}(s) = p(s) + u(s)$$

$p(s)$ the efficient log-price process, $u(s)$ microstructure noise component.
The one period, say $[t-1, t]$, integrated volatility is

$$IV(t) := \int_{t-1}^{t} \sigma^2(s)ds$$

Choose h step size ($h \to 0$)
$\widetilde{RV}^h(t)$ the realized volatility and $\widetilde{FM}^h(t)$ the Fourier estimator **in the presence of microstructure noise**

Studying R^2

Under the no leverage hypothesis:

$$Cov(IV(t+1), \widetilde{RV}^h(t)) = Cov(IV(t+1), RV^h(t)) = Cov(IV(t+1), IV(t))$$

Thus we compare:

$$R^2_{\widetilde{RV}} := \frac{Cov(IV(t+1), \widetilde{RV}_n(t))^2}{Var[IV(t)]Var[\widetilde{RV}_n(t)]} = \frac{Cov(IV(t+1), IV(t))^2}{Var[IV(t)]Var[\widetilde{RV}_n(t)]} \quad (4)$$

$$R^2_{\widetilde{FM}} := \frac{Cov(IV(t+1), \widetilde{FM}_{n,N}(t))^2}{Var[IV(t)]Var[\widetilde{FM}_{n,N}(t)]} = \frac{Cov(IV(t+1), IV(t))^2}{Var[IV(t)]Var[\widetilde{FM}_{n,N}(t)]} \quad (5)$$

Formulas (4) and (5) $\Rightarrow$ maximizing the R^2 of the linear regression is equivalent to minimizing the variance of the considered estimator $\Rightarrow$ minimum variance estimators should have better forecasting performances

Direct comparison of the forecasting performance of the two methodologies is possible by comparing the variance formulae varying the microstructure parameters

$$Var[\widetilde{RV}^h(t)] = Var[RV^h(t)] + 2V_u^2(\frac{2K_u}{h} - K_u + 1 + 4\frac{E[IV(t)]}{V_u}), \qquad (6)$$

(V_u = variance of the noise, K_u= kurtosis). As h (step size) $\to 0$ (6) diverges.

$$Var[\widetilde{FM}^{h,N}(t)] = Var[FM^{h,N}(t)] + \frac{2\pi}{h}\beta(h,N) + \gamma(h,N), \qquad (7)$$

where

$$\beta(h,N) := 4K_u V_u^2(1 + D_N^2(h) - 2D_N(h)),$$
$$\gamma(h,N) := 8V_u E[IV(t)] + 2V_u^2 - 2K_u V_u^2 + 4V_u^2(1+K_u)(2D_N(h) - D_N^2(h)),$$

and $D_N(h)$ is the Dirichlet kernel. Under the condition $hN^2 \to 0$:

$$\lim_{h\to 0, N\to\infty} \frac{2\pi}{h}\beta(h,N) = 0, \qquad \lim_{h\to 0, N\to\infty} \gamma(h,N) = 8V_u E[IV_t] + 2K_u V_u^2 + 6V_u^2$$

$\Rightarrow$ (7) converges to a constant.

Finally, given a sampling frequency h, it is possible to determine the optimal (in the sense of minimizing the variance) Fourier cutting frequency N_{cut}, having good effects on the forecasting performance.

Realized volatility type measures and the Fourier estimator: forecasting the integrated volatility one step (day) ahead
Three different data generating processes (as in [Andersen et al., 2006]):
M1) - GARCH Model

$$d\sigma^2(t) = \kappa(\theta - \sigma^2(t))dt + \overline{\sigma}\sigma^2(t)dW^2(t)$$

with $\kappa = 0.035$, $\theta = 0.636$, $\overline{\sigma} = 0.1439$, note that $E[IV(t)] = \theta = 0.636$,
M2) - Two-Factor Affine

$$\sigma^2(t) = \sigma_1^2(t) + \sigma_2^2(t), \quad d\sigma_i(t) = \kappa_i(\theta_i - \sigma_i^2(t))dt + \overline{\sigma}_i\sigma_i^2(t)dW^2(t) \quad i = 1, 2$$

with $\kappa_1 = 0.5708$, $\theta_1 = 0.3257$, $\overline{\sigma}_1 = 0.2286$, $\kappa_2 = 0.0757$, $\theta_2 = 0.1786$, and $\overline{\sigma}_2 = 0.1096$, note that $E[IV(t)] = \theta_1 + \theta_2 = 0.5043$,
M3) - Log-Normal Diffusion

$$d\log\sigma^2(t) = \kappa(\theta - \log\sigma^2(t))dt + \overline{\sigma}dW^2(t)$$

with $\kappa = 0.0136$, $\theta = -0.8382$, $\overline{\sigma} = 0.1148$, note that $E[IV(t)] = e^{\theta + \frac{\sigma^2}{4\kappa}} = 0.5510$.

Data generation

A simple Euler Monte Carlo discretization procedure generates high frequency evenly sampled theoretical prices $p(t)$ and observed returns by simulating **second-by-second return and variance** paths over $K = 240$ trading days (one trading year). A trading day is $T = 6$ hours for a total of $21,600$ observations (a tick corresponds to a second).

Then we sample the observations varying the uniform sampling interval $\rho(n) = \frac{T}{n}$ obtaining data sets with different frequencies. The initial point of the simulation of the volatility process is set at $E[IV(t)]$.

For each observation t_j the observed asset price is obtained by adding the microstructure noise component to the theoretical price: realizations $\eta(t_j)$ $(j = 0, 1, \ldots, n)$ come from a sequence of i.i.d. random variables with zero mean and constant variance.

Microstructure noise variance is set equal to a given percentage of the integrated volatility:

$$Var[\eta(t)] = \lambda E[IV(t)] \quad \text{with} \quad \lambda = 0\%,\ 0.1\%,\ 0.5\%.$$

We consider the following sampling periods:

n	2160	1440	720	360	288	96	48	24	12
$\rho(n)$	10"	15"	30"	1'	1' 15"	3'45"	7'30"	15'	30'

Forecasting future integrated volatility

Given the volatility process of the theoretical asset price, we calculate the exact integrated volatility:

The comparison between the realized and the Fourier volatility methodology is accomplished through the R^2 associated with the Mincer-Zarnowitz style linear regression of the integrated volatility at date $t+1$ ($IV(t+1)$) onto a constant and the integrated volatility of the previous day computed according to the realized volatility method and the Fourier method, formulae (4) and (5)

		M1			M2			M3		
λ	**n**	$R^2(RV)$	$R^2(FM_N)$	N	$R^2(RV)$	$R^2(FM_N)$	N	$R^2(RV)$	$R^2(FM_N)$	N
0%	**2160**	0,9035	0,9053	581	0,5707	0,5701	806	0,9010	0,9013	788
	1440	0,9028	0,9036	536	0,5575	0,5612	458	0,8914	0,8918	533
	720	0,8998	0,9033	326	0,5551	0,5579	335	0,8771	0,8806	345
	360	0,8919	0,8926	178	0,5097	0,5112	178	0,8592	0,8592	180
	288	0,8837	0,8862	124	0,5036	0,5049	144	0,8326	0,8349	143
	96	0,8025	0,8044	42	0,3846	0,3908	46	0,7270	0,7273	48
	48	0,7336	0,7344	24	0,3197	0,3170	24	0,6148	0,6170	24
	24	0,5750	0,5792	12	0,2176	0,2134	12	0,4571	0,4652	11
	12	0,4476	0,4538	5	0,1619	0,1709	5	0,3466	0,3315	6
0.1%	**2160**	0,6931	0,8861	170	0,3995	0,5377	251	0,8380	0,8771	386
	1440	0,7533	0,8763	143	0,4060	0,5232	218	0,8397	0,8639	267
	720	0,8103	0,8616	182	0,5040	0,5154	176	0,8413	0,8464	182
	360	0,8238	0,8435	121	0,4387	0,4601	155	0,8311	0,8336	180
	288	0,8085	0,8296	82	0,4238	0,4410	99	0,8216	0,8217	139
	96	0,7476	0,7506	42	0,3369	0,3536	37	0,7020	0,7007	47
	48	0,6633	0,6595	23	0,2876	0,2929	23	0,6027	0,6016	23
	24	0,5456	0,5459	12	0,2020	0,1975	12	0,4495	0,4603	11
	12	0,4244	0,4301	5	0,1605	0,1672	5	0,3433	0,3306	5
0.5%	**2160**	0,2224	0,8422	86	0,0878	0,4402	80	0,4317	0,8099	140
	1440	0,2249	0,8211	62	0,1086	0,3972	61	0,4415	0,7888	125
	720	0,3439	0,8019	50	0,1850	0,3790	51	0,6434	0,7486	98
	360	0,3776	0,7419	50	0,1973	0,3500	35	0,6667	0,7222	75
	288	0,4837	0,7099	31	0,2042	0,2749	32	0,6128	0,6683	60
	96	0,5254	0,6100	17	0,2429	0,2669	30	0,6021	0,6185	24
	48	0,4857	0,5757	18	0,2341	0,2395	24	0,5679	0,5719	23
	24	0,4650	0,4675	10	0,1550	0,1536	5	0,4139	0,4202	11
	12	0,3696	0,3907	4	0,1399	0,1450	5	0,3244	0,3088	6

Comments

- Concerning the **realized volatility** estimator, the R^2 goes up monotonically as the sampling horizon decreases only in a model without noise, if noise is added then the R^2 reaches the highest value for a sampling horizon between 1-5 minutes
- For the **Fourier estimator** the R^2 increases with the sampling frequency also in a model with microstructure noise
- The forecasting performance of the two estimators is quite similar in a model without noise. When noise is added the Fourier estimator outperforms the realized volatility estimator in a significant extent in particular for high frequency observations and when the noise component is relevant.
- When the noise increases, even maintaining the same size of the grid, the **cutting frequency** of the Fourier estimator becomes smaller and smaller: cutting the highest frequencies in the Fourier expansion we ignore high-frequency noise, as remarked in [M. and Sanfelici, 2008].

Forecasting: noise robust estimators

Compare the quality of the forecasts obtained through the Fourier method with that obtained by variants of the realized volatility estimator $RM_n(t)$ that turn out to be **robust** to microstructure noise, combining a fast time grid with some other slow time grids [Andersen et al., 2006]

- *sparse estimator* which is the equally spaced 75 second grid subsampled from the principal one
- *average estimator* by [Zhang et al., 2005]
- *Two Scaled estimator* combination of the average estimator with the classic realized volatility over the principal grid
- *Two Scaled Adjusted estimator* [Zhang 2006]
- [Zhou, 1996] estimator which extends the classical realized volatility by considering also the first-order serial correlation of high frequency returns
- *Kernel estimator* in [Barndorff-Nielsen and al., 2008a] (here Tukey-Hanning kernel)

		M1		M2		M3	
λ	**Model**	R^2	N	R^2	N	R^2	N
0.1%	RV_t^{all}	0,7533		0,3995		0,8397	
	RV_t^{sparse}	0,8085		0,4238		0,8311	
	$RV_t^{average}$	0,8796		0,5084		0,8581	
	RV_t^{TS}	0,8703		0,4853		0,8417	
	$RV_t^{TS_Adj}$	0,8703		0,4853		0,8417	
	RV_t^{Zhou}	0,7830		0,4512		0,8254	
	RV_t^{Ker}	0,8610		0,4977		0,8464	
	FM_t^N	0,8763	143	0,5232	218	0,8639	182
0.5%	RV_t^{all}	0,2249		0,0878		0,4415	
	RV_t^{sparse}	0,3837		0,2042		0,6128	
	$RV_t^{average}$	0,7608		0,3928		0,8022	
	RV_t^{TS}	0,7391		0,3598		0,7886	
	$RV_t^{TS_Adj}$	0,7391		0,3598		0,7886	
	RV_t^{Zhou}	0,2251		0,1005		0,5059	
	RV_t^{Ker}	0,6043		0,2516		0,7498	
	FM_t^N	0,8211	62	0,3972	61	0,7888	98

Tabella: R^2 for Integrated Variance Forecasts: linear regression of the integrated volatility at time $t+1$ onto a constant and the volatility at time t computed as a realized type volatility estimator or as the Fourier volatility.

Comment

Fourier method tends to prevail when the the noise level is high

Conclusion

We have seen that the Fourier estimator of covariance is:
(i) consistent under asynchronous trading,
(ii) positive definite,
(iii) asymptotically unbiased in the presence of various types of microstructure noise,
(iv) inconsistent in the presence of microstructure noise, nevertheless the MSE of the Fourier estimator converges to a constant as the number of observations increases
(v) further it allows us to treat volatility as an observable variable, thus we can exploit the knowledge of its path
(vi) has competitive forecasting performance in particular for high frequency observations and when the noise component is relevant
$\Rrightarrow$ a very interesting alternative especially when microstructure effects are particularly relevant in the available data

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